American Journal of Interdisciplinary Research and Development
ISSN Online: 2771-8948
Website: www.ajird.journalspark.org
Volume 11, Dec., 2022

ASYMPTOTIC EVALUATION OF PARAMETER-DEPENDENT INTEGRALS

Umirzakova Iroda
3rd Year Student of the Faculty of Mathematics of Samarkand State University named after Sharof Rashidov

Komilov Abdulaziz
3rd Year Student of the Faculty of Mathematics of Samarkand State University named after Sharof Rashidov

Abstract

This thesis presents the methods of calculating integrals that are approximate, but the calculation of their value is quite complicated.

Keywords: parameter-dependent integral, asymptotic estimation, Taylor series, Euler integrals.

When constructing mathematical models of life problems, in most cases, it is important to know the approximate value of the solution closest to this solution, rather than finding the exact value or values of the problem solution. This thesis considers the issue of approximate calculation of integrals depending on the parameter or asymptotic estimation of the parameter, which plays an important role in solving this type of problems.

In mathematics, an analytic function is a function that is locally given by a convergent power series.

First, we present the following theorem:

Theorem. If every term of the series $\sum_{n=1}^{\infty} u_n(x)$ is continuous in the segment $[a, b]$, and this series is uniformly convergent in this segment and $S(x)$ is a sum of series, then the following equality holds:

$$\int_a^b S(x)\,dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx.$$

Let the following integral depending on parameters p and q be given:

$$\Phi(p, q) = \int_a^b f(x, p) \cdot g(x, q)\,dx \qquad (1)$$

Let us assume that the functions $f(x, p)$, $g(x, q)$ are analytic in some interval $(-\rho, \rho)$, and this condition $(a, b) \subset (-\rho, \rho)$ is hold. Then the following equations hold for $\forall x \epsilon (a, b)$:

$$f(x,p)=\sum_{n=0}^{\infty}\frac{f^n(0,p)}{n!}\cdot x^n \quad g(x,q)=\sum_{n=0}^{\infty}\frac{f^n(0,q)}{n!}\cdot x^n$$

Using these equations and the above theorem, we can write integral (1) in the following form:

$$\Phi(p,q)=\int_a^b\sum_{n=0}^{\infty}\frac{f^n(0,p)}{n!}\cdot x^n\cdot g(x,q)\,dx=\sum_{n=0}^{\infty}\frac{f^n(0,p)}{n!}\int_a^b g(x,q)\cdot x^n\,dx,$$

$$\Phi(p,q)=\int_a^b\sum_{n=0}^{\infty}\frac{g^n(0,q)}{n!}\cdot x^n\cdot f(x,p)\,dx=\sum_{n=0}^{\infty}\frac{g^n(0,q)}{n!}\int_a^b f(x,p)\cdot x^n\,dx.$$

Here $\frac{f^n(0,p)}{n!}$ and $\frac{g^n(0,q)}{n!}$ are the Taylor coefficients of the functions $f(x,p)$ and $g(x,q)$, respectively.

For convenience, we choose one of the integrals $\int_a^b x^n f(x,p)\,dx$ and $\int_a^b x^n g(x,q)\,dx$ in such a way that to calculate the chosen integral be easy. Suppose that the equality$\int_a^b x^n f(x,p)\,dx=\psi(n,p)$ is hold, then

$$\int_a^b f(x,p)\cdot g(x,q)\,dx=\int_a^b\sum_{n=0}^{\infty}\frac{g^n(0,q)}{n!}\cdot x^n f(x,p)\,dx=$$

$$=\sum_{n=0}^{\infty}\frac{g^n(0,q)}{n!}\int_a^b f(x,p)\cdot x^n\,dx=\sum_{n=0}^{\infty}\frac{g^n(0,q)}{n!}\cdot\psi(n,p).$$

If we choose $n<n_0$ in this equation, we arrive at the following approximate equation:

$$\int_a^b f(x,p)\cdot g(x,q)\,dx\approx\sum_{n=0}^{n_0}\frac{g^n(0,q)}{n!}\cdot\psi(n,p)$$

It is known that the larger the number n_0 can be, the closer the found value is to the value of the given integral.

Some methods of calculating integrals depending on parameter are given below:

Problem 1. Evaluate the value of the following integral: $I(r)=\int_0^{\pi}e^{mr\cdot\cos x}\,dx.$

Solution. First, we split this integral into two integrals, and then get the substitution:

$$I(r)=\int_0^{\pi}e^{mr\cdot\cos x}\,dx=\int_0^{\frac{\pi}{2}}e^{mr\cdot\cos x}\,dx+\int_{\frac{\pi}{2}}^{\pi}e^{mr\cdot\cos x}\,dx=\begin{bmatrix} x=t+\frac{\pi}{2} \\ dx=dt \\ x=\frac{\pi}{2}\to t=0 \\ x=\pi\to t=\frac{\pi}{2}\end{bmatrix}=$$

$$=\int_0^{\frac{\pi}{2}} e^{mr\cdot\cos x}\,dx+\int_0^{\frac{\pi}{2}} e^{mr\cdot\cos\left(t+\frac{\pi}{2}\right)}\,dt=\int_0^{\frac{\pi}{2}} e^{mr\cdot\cos x}\,dx+\int_0^{\frac{\pi}{2}} e^{-mr\cdot\sin x}\,dx.$$

We can substitute in the integrals on the right side of the last equality:

$$\int_0^{\frac{\pi}{2}} e^{mr\cdot\cos x}\,dx=[\cos x=t]=\int_0^1\frac{e^{mrt}}{\sqrt{1-t^2}}dt,$$

$$\int_0^{\frac{\pi}{2}} e^{-mr\cdot\sin x}\,dx=[\sin x=z]=\int_0^1\frac{e^{-mrz}}{\sqrt{1-z^2}}dz=\int_0^1\frac{e^{-mrt}}{\sqrt{1-t^2}}dt.$$

Then,

$$I(r)=\int_0^1\frac{e^{mrt}+e^{-mrt}}{\sqrt{1-t^2}}dt.$$

Now we use this expansion: $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$. Then we arrive at the following equality:

$$I(r)=\int_0^1\left(\frac{1}{\sqrt{1-t^2}}\sum_{n=0}^{\infty}\frac{(mrt)^n+(-mrt)^n}{n!}\right)dt.$$

It is known that in the sum on the right side of the above equation, the odd-numbered terms become zero. So,

$$I(r)=2\int_0^1\sum_{n=0}^{\infty}\frac{(m\cdot r)^{2n}t^{2n}}{2n!\cdot\sqrt{1-t^2}}dt=\sum_{n=0}^{\infty}\frac{(mr)^{2n}}{(2n)!}\int_0^1\frac{2t^{2n}}{\sqrt{1-t^2}}dt=\begin{bmatrix} t^2=x \\ t=\sqrt{x} \\ dt=\dfrac{1}{2\sqrt{x}}dx\end{bmatrix}=$$

$$=2\sum_{n=0}^{\infty}\frac{(mr)^{2n}}{(2n)!}\int_0^1\frac{x^n}{(1-x)^{\frac{1}{2}}}\cdot\frac{1}{2\sqrt{x}}dx=\sum_{n=0}^{\infty}\frac{(mr)^{2n}}{(2n)!}\int_0^1 x^{n-\frac{1}{2}}\cdot(1-x)^{-\frac{1}{2}}dx.$$

The integral on the right side of the resulting equation can be calculated using the beta function:

$$I(r)=\sum_{n=0}^{\infty}\frac{(mr)^{2n}}{(2n)!}\cdot B\left(n+\frac{1}{2};\frac{1}{2}\right)=\sum_{n=0}^{\infty}\frac{(mr)^{2n}}{(2n)!}\cdot\frac{\Gamma\left(n+\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma(n+1)}=$$

$$=\sum_{n=0}^{\infty}m^{2n}r^{2n}\cdot\frac{\pi(2n-1)!!}{2^n\cdot n!\cdot(2n)!}=\sum_{n=0}^{\infty}\frac{m^{2n}r^{2n}\pi}{2^{2n}(n!)^2}.$$

Therefore,

$$I(r)=\sum_{n=0}^{\infty}\frac{m^{2n}r^{2n}\pi}{2^{2n}(n!)^2}.$$

Problem 2. Evaluate the value of the following integral: $I(b)=\int_0^{+\infty}e^{-ax^2}\cos bx\,dx.$

Solution.

It is known that the given integral depending on the parameter is uniformly convergent according to the comparison property. So, the sign of integral can be replaced by the sign of sum. To calculate the value of this integral, we use the following Taylor expansion: $\cos bx = \sum_{n=0}^{\infty}\frac{(-1)^n(bx)^{2n}}{(2n)!}$. In that case,

$$I(b) = \int_0^{+\infty} e^{-ax^2}\sum_{n=0}^{\infty}\frac{(-1)^n(bx)^{2n}}{(2n)!}dx = \sum_{n=0}^{\infty}\int_0^{+\infty} e^{-ax^2}\frac{(-1)^n(bx)^{2n}}{(2n)!}dx =$$

$$= \begin{bmatrix} ax^2 = t \\ x = \sqrt{\frac{t}{a}} \\ dx = \frac{1}{2\sqrt{at}}dt \end{bmatrix} = \sum_{n=0}^{\infty}\frac{(-1)^n b^{2n}}{(2n)!}\int_0^{+\infty} e^{-t}\left(\sqrt{\frac{t}{a}}\right)^{2n}\left(\frac{1}{2\sqrt{at}}\right)dt =$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n b^{2n}}{2(2n)!\,a^{n+\frac{1}{2}}}\int_0^{+\infty} e^{-t}t^{n-\frac{1}{2}}dt = \sum_{n=0}^{\infty}\frac{(-1)^n b^{2n}}{2(2n)!\,a^{n+\frac{1}{2}}}\Gamma\left(n+\frac{1}{2}\right) =$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n b^{2n}}{2(2n)!\,a^{n+\frac{1}{2}}}\cdot\frac{(2n-1)!!\sqrt{\pi}}{2^n} = \frac{1}{2}\sqrt{\frac{\pi}{a}}\sum_{n=0}^{\infty}\frac{(-1)^n b^{2n}}{2^n\cdot(2n)!!\,a^n} = \frac{1}{2}\sqrt{\frac{\pi}{a}}\sum_{n=0}^{\infty}\frac{(-1)^n b^{2n}}{4^n a^n\cdot n!}.$$

So, the following equality is hold for the given integral:

$$I(b) = \frac{1}{2}\sqrt{\frac{\pi}{a}}\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{n!}\cdot\left(\frac{b^2}{4a}\right)^n\right].$$

It is not difficult to know that the right side of this equation is the Taylor expansion of the function $f(b) = e^{-\frac{b^2}{4a}}$. So the value of the given integral is equal to:

$$u(b) = \int_0^{+\infty} e^{-ax^2}\cos bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\sum_{n=0}^{\infty}\frac{(-1)^n b^{2n}}{4^n a^n\cdot n!} = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-\frac{b^2}{4a}}.$$

Problem 3. Evaluate the value of the following integral: $\int_0^b f(x)(b-x)^\alpha\,dx$.

Here, $\alpha > -1$ and $f(x)$ is an analytic function.

Solution.

Since $\alpha > -1$ and $f(x)$ is an analytic function in the given integral, the integral depending on this parameter is convergent (Abel's sign). Also, the function $f(x)$ can be expanded into a power series. Thus

$$f(x) = \sum_{n=1}^{\infty}\frac{a_n x^n}{n!}.$$

here $a_n -$ coefficients of extension. Then

$$\int_0^b f(x)(b-x)^\alpha\,dx = \int_0^b \sum_{n=1}^{\infty} \frac{a_n x^n}{n!}(b-x)^\alpha\,dx = \sum_{n=0}^{\infty} \frac{a_n}{n!}\int_0^b x^n(b-x)^\alpha\,dx =$$

$$= \begin{bmatrix} x = bt \\ dx = bdt \\ x = b \to t = 1 \\ x = 0 \to t = 0 \end{bmatrix} = \sum_{n=1}^{\infty} \frac{a_n}{n!}\int_0^1 b^n t^n (b-bt)^\alpha\,bdt = \sum_{n=1}^{\infty} \frac{a_n b^{n+\alpha+1}}{n!}\int_0^1 t^n(1-t)^\alpha\,dt =$$

$$= \sum_{n=1}^{\infty} \frac{a_n b^{n+\alpha+1}}{n!} B(n+1;\alpha+1) = \sum_{n=1}^{\infty} \frac{a_n b^{n+\alpha+1}}{n!}\cdot\frac{\Gamma(n+1)\cdot\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)} =$$

$$= \sum_{n=1}^{\infty} \frac{a_n b^{n+\alpha+1}\cdot\alpha\Gamma(\alpha)}{(n+\alpha+2)(n+\alpha)\Gamma(n+\alpha)}.$$

Problem 4. Evaluate the value of the following integral: $I(\alpha) = \int_0^1 \sqrt{1-x}\sin ax\,dx$.

Solution. To calculate this integral, we use the following Taylor expansion:

$\sin \alpha x = \sum_{n=1}^{\infty} \frac{(-1)^n \alpha^{2n+1} x^{2n+1}}{(2n+1)!}$. Then,

$$I(\alpha) = \int_0^1 \sqrt{1-x}\cdot\sin ax\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1} x^{2n+1}}{(2n+1)!}\sqrt{1-x}\,dx =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1}}{(2n+1)!}\int_0^1 x^{2n+1}(1-x)^{\frac{1}{2}}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1}}{(2n+1)!}\cdot B\left(2n+2,\frac{3}{2}\right) =$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1}}{(2n+1)!}\cdot\frac{\Gamma(2n+2)\cdot\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(2n+\frac{7}{2}\right)} = \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1}}{(2n+1)!}\cdot\frac{(2n+1)!\cdot\frac{1}{2}}{\frac{\left((2\cdot(2n+2))!!\sqrt{\pi}\right)}{2^{2n+2}}} =$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n\cdot\alpha^{2n+1}\cdot 2^{2n+2}}{2^{2n+1}\cdot n!\sqrt{\pi}} = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty} \frac{(-1)^n\cdot\alpha^{2n+1}}{n!}.$$

Problem 5. Evaluate the value of the following integral: $I(\alpha) = \int_0^1 x^{\alpha x}\,dx$.

Solution.

To calculate this integral, we first get the following substitution:

$$I(\alpha) = \int_0^1 x^{\alpha x}\,dx = \begin{bmatrix} x = e^{-\xi} \\ dx = -e^{-\xi}d\xi \\ x \to 0 \quad \xi \to \infty \\ x \to 1 \quad \xi \to 0 \end{bmatrix} = \int_0^{+\infty} e^{-\xi}\cdot e^{-\xi\alpha e^{-\xi}}\,d\xi.$$

Now we use the following Taylor expansion: $e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$. Then,

$$I(\alpha)=\int_0^{+\infty} e^{-\xi}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\alpha\xi e^{-\xi}\right)^n}{n!}d\xi=\sum_{n=0}^{\infty}\frac{(-1)^n\alpha^n}{n!}\int_0^{+\infty}\xi^n e^{-\xi(n+1)}\,d\xi=$$

$$=\begin{bmatrix}\xi(n+1)=\psi\\ d\xi=\dfrac{d\psi}{n+1}\end{bmatrix}=\sum_{n=0}^{\infty}\frac{(-1)^n\alpha^n}{n!}\int_0^{+\infty}\frac{\left(\dfrac{\psi}{n+1}\right)^n e^{-\psi}}{n+1}d\psi=$$

$$=\sum_{n=0}^{+\infty}\frac{(-1)^n\alpha^n}{n!\,(n+1)^{n+1}}\int_0^{+\infty}\psi^n e^{-\psi}\,d\psi=\sum_{n=0}^{\infty}\frac{(-1)^n\alpha^n}{n!\,(n+1)^{n+1}}\Gamma(n+1)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\alpha^n}{n^n}.$$

Thus, following equality is holds for given integral:

$$I(\alpha)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\alpha^n}{n^n}.$$

In particular, $\int_0^1 x^x\,dx=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^n}$ $for\ \alpha=1$, and $\int_0^1\frac{1}{x^x}dx=\sum_{n=1}^{\infty}\frac{1}{n^n}$ $for\ \alpha=1$.

Problem 6. Evaluate the value of the following integral: $\psi(\alpha)=\int_0^{+\infty}e^{-\alpha x^2+x}\sin x$.

Solution. It is known that the given integral depending on the parameter is uniformly convergent for $\alpha>0$ according to the comparison property. So, the sign of integral can be replaced by the sign of sum. To calculate the value of this integral, we use the following Taylor expansion: $e^x\sin x=\sum_{n=0}^{\infty}\frac{2^{\frac{n}{2}}\sin\frac{n\pi}{4}}{n!}\cdot x^n$. In that case,

$$\psi(\alpha)=\int_0^{+\infty}e^{-\alpha x^2}\sum_{n=0}^{\infty}\frac{2^{\frac{n}{2}}\sin\dfrac{n\pi}{4}}{n!}\cdot x^n\,dx=\sum_{n=0}^{+\infty}\frac{2^{\frac{n}{2}}\sin\dfrac{n\pi}{4}}{n!}\int_0^{+\infty}e^{-\alpha x^2}\cdot x^n\,dx=$$

$$=\begin{bmatrix}\alpha x^2=\xi\\ x=\sqrt{\dfrac{\xi}{\alpha}}\\ dx=\dfrac{1}{2\sqrt{\xi\alpha}}d\xi\end{bmatrix}=\sum_{n=0}^{+\infty}\frac{2^{\frac{n}{2}}\sin\dfrac{n\pi}{4}}{n!}\int_0^{+\infty}e^{-\xi}\left(\frac{\xi}{\alpha}\right)^{\frac{n}{2}}\cdot\frac{1}{2\sqrt{\xi\alpha}}d\xi=$$

$$=\frac{1}{2}\sum_{n=0}^{+\infty}\frac{2^{\frac{n}{2}}\sin\dfrac{n\pi}{4}}{n!\,\alpha^{\frac{n+1}{2}}}\int_0^{+\infty}e^{-\xi}\cdot\xi^{\frac{n-1}{2}}d\xi=\frac{1}{2}\sum_{n=0}^{+\infty}\frac{2^{\frac{n}{2}}\sin\dfrac{n\pi}{4}}{n!\,\alpha^{\frac{n+1}{2}}}\Gamma\left(\frac{n+1}{2}\right).$$

Probem 7. Evaluate the value of the following integral: $I(\beta)=\int_0^{+\infty}\frac{e^{-\alpha x}}{\beta+x}dx$.

Solution. It is known that the given integral depending on the parameter is uniformly convergent according to the comparison property. So, the sign of integral can be replaced by the sign of sum. To calculate the value of this integral, we use the following Taylor expansion: $\frac{1}{1+\beta x} = \sum_{n=0}^{\infty}(-1)^n\beta^n x^n$. In that case,

$$I(\beta) = \int_0^{+\infty} e^{-\alpha x}\cdot\sum_{n=0}^{\infty}(-1)^n\beta^n x^n\,dx = \sum_{n=0}^{\infty}(-1)^n\beta^n\int_0^{+\infty} x^n e^{-\alpha x}\,dx = \begin{bmatrix}\alpha x = \phi \\ dx = \dfrac{d\phi}{\alpha}\end{bmatrix} =$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n\beta^n}{\alpha^{n+1}}\int_0^{+\infty}\phi^n e^{-\phi}\,d\phi = \sum_{n=0}^{\infty}\frac{(-1)^n\beta^n}{\alpha^{n+1}}\Gamma(n+1) = \sum_{n=0}^{\infty}\frac{(-1)^n\beta^n n!}{\alpha^{n+1}}.$$

Thus,

$$I(\beta) = \sum_{n=0}^{\infty}\frac{(-1)^n\beta^n n!}{\alpha^{n+1}}.$$

Problem 8. Evaluate the value of the following integral: $\int_0^{+\infty} e^{-\alpha x^p}\sin\beta x^q\,dx$, here $p > 0$, $q > -1$.

Solution. It is known that the given integral depending on the parameter is uniformly convergent for $a > 0$ according to the comparison property. So, the sign of integral can be replaced by the sign of sum. To calculate the value of this integral, we use the following Taylor expansion: $\sin x = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}$. In that case,

$$\int_0^{+\infty} e^{-\alpha x^p}\sin\beta x^q\,dx = \int_0^{+\infty} e^{-\alpha x^p}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\beta^{2n-1}x^{q(2n-1)}}{(2n-1)!}dx =$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\beta^{2n-1}}{(2n-1)!}\int_0^{+\infty} e^{-\alpha x^p}x^{q(2n-1)}\,dx = \begin{bmatrix}\alpha x^p = \xi \\ x = \sqrt[p]{\dfrac{\xi}{\alpha}} \\ dx = \xi^{\frac{1}{p}-1}\cdot\dfrac{1}{p\cdot\alpha^{\frac{1}{p}}}d\xi\end{bmatrix} =$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\beta^{(2n-1)}}{(2n-1)!\cdot p\alpha^{\frac{q(2n-1)+1}{p}}}\int_0^{+\infty} e^{-\xi}\xi^{\frac{q(2n-1)+1}{p}-1}\,dx =$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\beta^{(2n-1)}}{(2n-1)!\,p\alpha^{\frac{q(2n-1)+1}{p}}}\Gamma\left(\frac{q(2n-1)+1}{p}\right).$$

In this case, if $\alpha = 1, \beta = 1, p = 2, q = 4$, we get the following equation:

$$\int_0^{+\infty} e^{-x^2}\sin x^4\,dx = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\beta^{2n-1}}{(2n-1)!\,2\alpha^{\frac{(4(2n-1)+1)}{2}}}\Gamma\left(\frac{4(2n-1)+1}{2}\right) =$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\beta^{2n-1}}{2(2n-1)!\,\alpha^{4n-\frac{3}{2}}}\Gamma\left((4n-2)+\frac{1}{2}\right)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\beta^{2n-1}}{2(2n-1)!\,\alpha^{4n-\frac{3}{2}}}\cdot\frac{(2(4n-2)-1)!!\,\sqrt{\pi}}{2^{4n-2}}$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\beta^{2n-1}}{2(2n-1)!\,\alpha^{4n-\frac{3}{2}}}\cdot\frac{(8n-3)!!\,\sqrt{\pi}}{2^{n}}.$$

Problem 9. Evaluate the value of the following integral: $I(p)=\int_0^{\infty}\frac{e^{-\alpha x^p}}{x^2+bx+c}dx.$

Solution. It is known that the given integral depending on the parameter is uniformly convergent for $a>0$ according to the comparison property. So, the sign of integral can be replaced by the sign of sum. To calculate the value of this integral, we use the following Taylor expansion: $\frac{1}{x^2+bx+c}=\frac{1}{\sqrt{c}}\sum_{n=0}^{\infty}\frac{x^n\sin(n+1)\phi}{\sin\phi}$,

here $\phi=-\arctan\sqrt{\frac{4c}{b^2}-1}$.

$$\int_0^{\infty}\frac{e^{-\alpha x^p}}{x^2+bx+c}dx=\int_0^{+\infty}e^{-\alpha x^p}\sum_{n=0}^{\infty}\frac{x^n\sin(n+1)\phi}{c\cdot\sin\phi}=\frac{1}{c}\sum_{n=0}^{\infty}\frac{\sin(n+1)\phi}{\sin\phi}\int_0^{+\infty}e^{-\alpha x^p}x^n\,dx$$

$$=\left[\begin{array}{c}\alpha x^p=\xi\\ x=\sqrt[p]{\left(\frac{\xi}{\alpha}\right)}\\ dx=\frac{\xi^{\frac{1}{p}-1}}{p\alpha^{\frac{1}{p}}}d\xi\end{array}\right]=\frac{1}{c}\sum_{n=0}^{\infty}\frac{\sin(n+1)\phi}{\sin\phi}\int_0^{+\infty}e^{-\xi}\cdot\left(\frac{\xi}{\alpha}\right)^{\frac{n}{p}}\cdot\frac{\xi^{\frac{1}{p}-1}}{p\alpha^{\frac{1}{p}}}d\xi=$$

$$=\frac{1}{c}\sum_{n=0}^{\infty}\frac{\sin(n+1)\phi}{p\cdot\alpha^{\frac{n+1}{p}}\cdot\sin\phi}\int_0^{+\infty}e^{-\xi}\xi^{\frac{n+1}{p}-1}d\xi=\frac{1}{c}\sum_{n=0}^{\infty}\frac{\sin(n+1)\phi}{p\cdot\alpha^{\frac{n+1}{p}}\cdot\sin\phi}\Gamma\left(\frac{n+1}{p}\right).$$

Problem 10. Evaluate the value of the following integral: $E(k)=\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\phi}\,d\phi.$

Solution. To evaluate this integral, we use the following Taylor expansion

$\sqrt{1-x}=1-\sum_{n=1}^{\infty}\frac{(2n-3)!!}{2^n\cdot n!}x^n.$

In that case,

$$E(k)=\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\phi}\,d\phi=\int_0^{\frac{\pi}{2}}(1-\sum_{n=1}^{\infty}\frac{((2n-3)!!)}{2^n n!}\cdot(k^2\sin^2\phi)^n\,d\phi=$$

$$=\frac{\pi}{2}-\sum_{n=1}^{\infty}\frac{(2n-3)!!\,k^{2n}}{2^n n!}\int_0^{\frac{\pi}{2}}\sin^{2n}\phi\,d\phi=\left[\int_0^{\frac{\pi}{2}}\sin^{2n}\phi\,d\phi=\frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{2}\right]=$$

$$=\frac{\pi}{2}-\sum_{n=1}^{\infty}\frac{(2n-3)!!\,k^{2n}}{2^n n!}\cdot\frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{2}=\frac{\pi}{2}\left(1-\sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{n!}\right)^2\cdot\frac{k^{2n}}{2^{2n}(2n-1)}\right)=$$

$$=\frac{\pi}{2}\left(1-\sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2\cdot\frac{k^{2n}}{(2n-1)}\right).$$

If we take $k=\frac{1}{2}$ in the given integral, the following equality is holds:

$$\int_0^{\frac{\pi}{2}}\sqrt{1-\frac{1}{4}\sin^2 x}\,dx=\frac{\pi}{2}\left(1-\sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2\cdot\frac{1}{2^{2n}(2n-1)}\right).$$

Problem 11. Evaluate the value of the Bessel function: $I(x)=\frac{1}{\pi}\int_0^{\pi}\cos(x\cdot\sin\phi)\,d\phi$.

Solution. To evaluate this integral, we use the following Taylor expansion

$\cos x=\sum_{n=0}^{\infty}\frac{((-1)^n x^{2n})}{(2n)!}$.

In that case,

$$I(x)=\frac{1}{\pi}\int_0^{\pi}\cos(x\cdot\sin\phi)\,d\phi=\frac{1}{\pi}\int_0^{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n(x\cdot\sin\phi)^{2n}}{(2n)!}d\phi=$$

$$=\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\int_0^{\pi}\sin^{2\mathrm{n}}\phi\,d\phi$$

$$=\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\int_0^{\frac{\pi}{2}}\sin^{2n}\phi\,d\phi+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\int_{\frac{\pi}{2}}^{\pi}\sin^{2n}\phi\,d\phi.$$

In the second integral on the right side of the last equality, we can define it as $\phi=\frac{\pi}{2}+\psi$. In that case,

$$I(x)=\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\int_0^{\frac{\pi}{2}}\sin^{2n}\phi\,d\phi+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\int_0^{\frac{\pi}{2}}\cos^{2n}\psi\,d\psi=$$

$$=\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\int_0^{\frac{\pi}{2}}\sin^{2n}\phi\,d\phi=\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\cdot\frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{2}=$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\cdot\frac{(2n-1)!!}{(2n)!!}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\cdot\frac{(2n-1)!!}{2^n\cdot n!}.$$

Similar expansions can be successfully used to approximate integrals that cannot be expressed in finite form and to compile tables of their values.

References

1. Alimov Sh., Ashurov R. Matematik tahlil. 3-qism. “Mumtoz so`z”, Toshkent, 2018.
2. Демидович Б.П. Сборник задач и упражнений по математическому анализу. Издательство ЧеРо, 13-е издание. 1997, Москва.
3. Sa’dullayev A., Mansurov H., Xudoyberganov G. va b.q. Matematik analiz kursidan misol va masalalar to`plami. 3-qism. “O`zbekiston” nashriyoti. Toshkent, 1993.
4. Weisstein, Eric W. "Taylor Series". MathWorld.
5. Guoning Wu. Integrals Depending on a Parameter. China University of Petroleum-Beijing 2017.9.