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# ON τ-BOUNDED SPACES

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**Abstract**:
In this article, it has been proven that τ-bounded spaces, if they satisfy the T2 (Hausdorff space) condition, are also T3 (Regular space) but not necessarily T4 (Normal space), with a counterexample provided. Additionally, the relationship between local weak density and local density has been examined.

**Keywords**: τ-bounded space, T2 (Hausdorff space), T3 (Regular space), T4 (Normal space), local weak density, local density.

**Introduction**

Here τ is cardinal number.

**Definition 1.** A topological space $(X, \tau)$ is called $\tau$-bounded if the closure of any subset of the space, whose cardinality does not exceed $\tau$, is compact.

**Definition 2.** In topology, compactness means that from any open cover of a set, a finite subcover can be chosen.

**Definition 3.** A topological space is called $T_2$ (Hausdorff space) if, for any two distinct points, there exist non-intersecting open sets separating them. That is, for each $x \neq y$, there exist open sets $U$ and $V$ such that $x \in U$, $y \in V$ , and $U \cap V = \emptyset$.

**Definition 4.** A topological space is called $T_3$ (Regular space) if, for any closed set and a point not belonging to it, there exist disjoint open sets separating them. That is, for each $x \in U$ and $A \subseteq V$, and $U \cap V = \emptyset$ and $U$ and $V$ are open sets such that $V$ covers $A$.

**Definition 5.** A set $D \subseteq X$ is said to be dense around a point $x \in X$ if, for every open neighborhood $U$ of $x$, we have $U \cap D \neq \emptyset$. In other words, a set is locally dense if every open set contains at least one point from the dense set.

**Definition 6.** A set $A \subseteq X$ is said to be locally weakly dense around a point $x \in X$ if, for every open neighborhood of $x$, there exists a smaller dense subset.

**Theorem 1**: If $(X, \tau)$ is a $\tau$-bounded and $T_2$ (Hausdorff) space, then it is also $T_3$ (Regular). That is, for any closed set and a point outside it, there exist disjoint open sets separating them.

$\tau$-boundedness requires that the closure of any subset, whose cardinality does not exceed $\tau$, is compact. This means that if a set is limited in size, its closure will be compact. This property is crucial because compact sets and their characteristics allow the separation of closed sets and points in a $T_3$ space.

We know that one of the properties of Hausdorff spaces is that compact sets are closed, and they can also be separated from points. Therefore, any compact set and point can be separated.

Now, let is consider the space being $\tau$-bounded.

If we are given a closed set $A$ and a point $x \notin A$, we take the subset of $A$ whose cardinality does not exceed $\tau$ (or the set itself, if it is small). Since the space is $\tau$-bounded, the closure of this set will be compact.

Since the set is compact, we can use the Hausdorff property to find disjoint open sets separating $A$ and the point $x$. The Hausdorff property allows us to find open sets $U$ containing $x$ and $V$ containing $A$ such that:

$x \in U_x$,

$A \subseteq V_A$,

$U \cap V = \emptyset$

This satisfies the regularity ($T_3$) condition.

Thus, if a $\tau$-bounded space satisfies the $T_2$ (Hausdorff) condition, it will also satisfy the $T_3$ (Regular) condition, meaning any closed set and point outside it can be separated by disjoint open sets.

Next, we will consider a counterexample showing that a $\tau$-bounded space satisfying the $T_2$ (Hausdorff space) condition does not necessarily satisfy the $T_4$ (Normal space) condition.

Let us examine the product space $W \times W_0$ where $W_0$ is the space of all countable ordinal numbers, and $W$ is the space of all ordinal numbers less than or equal to $w_1$. It is known that $W$ and $W_0$ are τ-bounded and satisfy the $T_2$ (Hausdorff) condition, but their product $W \times W_0$ does not satisfy the Normal space condition.

Thus, a space satisfying the $T_2$ (Hausdorff) condition does not necessarily satisfy the $T_4$ (Normal space) condition.

**Theorem 2.** If $(X, \tau)$ is $\tau$-bounded space, then $ldX = lwdX$.

**Proof**: It is known that in a $\tau$-bounded space, the closure of any subset whose cardinality does not exceed $\tau$ is compact. This means that sets with limited cardinality in such spaces possess compactness properties.

Local weak density means that for any point $x \in X$, there exists a smaller dense subset around the point. This implies that for any open neighborhood $U$ of $x$, there exists a dense subset $D \subseteq U$.

Thus, if a set $A$ is locally weakly dense, we can find smaller dense subsets in every open neighborhood of any point. Now, we move on to proving local density. The meaning of local density is that within any open neighborhood, the set itself is dense. If $A \subseteq X$ is locally weakly dense, then within every open neighborhood, smaller dense subsets can be found. $\tau$-boundedness ensures that the closure of any subset with cardinality not exceeding $\tau$ is compact. Thus, if we find dense subsets within smaller parts of $A$, the closure of these subsets will be compact.

By finding dense parts within any open neighborhood, we ensure that $A$ itself is dense. This shows that local weak density actually guarantees local density because smaller dense subsets can be found within any open neighborhood, making the whole set dense.
Thus, $ldX = lwdX$.

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